THE FOUR QUADRANTS OF THE SVD AND ITS FOURIER TRANSFORM IN A TWO CHANNEL SYSTEM WITH MULTIPATH

Adam Bojanczyk, James LaRue**, Edmond Rusjan***, Irizarry A. Vega****, Andrew Williams*****

* Electrical and Computer Engineering, Cornell University, USA
**CACI at AFRL Information Directorate, Rome, New York, USA
*** Mathematics Department, SUNYIT, Utica New York, USA
****AFRL-IFEC Information Directorate, Rome, New York, USA

ABSTRACT

We address the problem of blind deconvolution in a two channel system by analyzing the singular vectors of the system. In particular, we study the singular value decomposition (SVD) of a typical two block system of convolution matrices made from data measured on two receivers. We focus on the singular vectors, rather than on the singular values. We discover that the singular vectors can be divided into four sub-structures. We conjecture that this four quadrant sub-structure separates the two channels into the two source signals and the two multipath functions. The analysis of the singular vector substructures in a very controlled simulation results in an improvement in determining the number of channel coefficients and a better understanding of channel characteristics.

1. INTRODUCTION

The goal of blind deconvolution is to estimate the number of channel coefficients and their values in order to construct a filter that will offset the effects of multipath and additive noise. Many algorithms have been designed that first calculate filter length based on the eigenvalue outputs [1]. Historically, in a high signal to noise (SNR) multi-channel system, the number of zero eigenvalues has been identified with the dimension of the nullspace which in turn is identified with the number of multipath coefficients. However, as the SNR is lowered, or a DC offset is added, the estimations fail due to the smoothing out of the eigenvalues at the transition from the nullspace to the signal subspace [2].

The singular value decomposition (SVD) algorithm has long been a workhorse for subspace techniques involving multiple channels. In our analysis, we exploit the relationship between the SVD of a matrix M and the eigenvalue decompositions (EVD) of M^TM and MM^T. The left singular vectors of the matrix M are the eigenvectors of the matrix M^TM, the right singular vectors of the matrix M are the eigenvectors of the matrix MM^T and the singular values of the matrix M are the square roots of the eigenvalues of (both) the matrices M^TM and MM^T. This relationship allows us to transparently move from SVD analysis to EVD analysis and back in our theoretical analysis. Numerically, of course, the SVD is superior, so we employ SVD in all the simulations and tests.

We use two other important tools in our theoretical analysis. The first is the beautiful connection between circulant matrices and the Discrete Fourier Transforms (DFT) – the DFT matrix diagonalizes circulant matrices. Furthermore, the DFT of the generating vector of the circulant matrix equals its spectrum, i.e., the sequence of its eigenvalues.

The second tool is the asymptotic equivalence of circulant and Toeplitz matrices. This allows us to approximate the matrices arising in our study, which are Toeplitz, by the corresponding circulant matrix. Consequently, we can transfer the powerful theoretical results, valid for circulant matrices, approximately (valid in an asymptotic way) to Toeplitz matrices.

Using a matrix formulation based on the convolutional model for two channels [3], the EVD output is analyzed in detail. We have found a subsystem of four quadrants of eigenvectors that make up the EVD; two quadrants relate to the multipath spaces of the two channels and two quadrants relate the signal subspace from each channel.

For a single channel, the DFT of a sequence is directly related to the EVD of its corresponding (usually full rank) circulant matrix [4]. We conjecture that this result can be extended to the case of multiple channels. We will demonstrate that the way eigenvalues are calculated is critical to getting out meaningful information from the so called nullspace especially in the presence of noise. Specifically, when there is no noise, multiple zero eigenvalues are in fact all equal to zero and the eigenvectors reflect this commonality when measured in either of the frequency or statistical domains [5]. In the case of a rank one nullspace it has been shown that the single nullvector matches the channel vector up to a phase ambiguity [6]. However, either adding noise or adding a DC offset to the received data before forming the two-channel data matrix changes the zero eigenvalues into a set of monotonically increasing non-zero eigenvalues. This apparently allows for a meaningful DFT to appear in the nullspace for each channel.

2. CIRCULANT MATRIX AND THE DFT

Starting with the sequence $x = \{x_1, x_2, x_3, \ldots, x_m\}$, let

$$C = \begin{bmatrix} x_1 & x_2 & \ldots & \ldots & x_m \\ x_m & x_1 & \ldots & \ldots & x_{m-1} \\ x_{m-1} & x_m & \ldots & \ldots & x_{m-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_2 & x_3 & \ldots & \ldots & x_1 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 1 & \ldots & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(m-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \omega^{m-1} & \omega^{2(m-1)} & \ldots & \omega^{(m-1)(m-1)} \end{bmatrix},$$

and $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m)$, where $\lambda_i = \sum_{k=0}^{m-1} x_k \omega^{ki} \quad and \quad \omega = e^{-2j\pi/m}$. C is a circulant matrix, F is a Fourier transform matrix whose columns form a very special basis and are eigenvectors of C. Each column is associated with a frequency in the DFT. Λ is the diagonal matrix of the DFT values of x. Through the spectral decomposition theorem, these DFT values give amplitude (measure) to the frequency basis functions. C can be written as $C = F \Lambda F^H$ where the H stands for hermitian. As already mentioned the eigendecomposition of a circulant matrix is equivalent to the DFT of the sequence of numbers that make up the circulant matrix.

3. FORMING THE BLOCK TOEPLITZ MATRIX

Multipath, for this paper, applies to a signal that has been corrupted by adding in either early or delayed copies of itself. One way to look at the problem is to model the received signal as a convolution process in the time domain.

$$y(t) = \sum_{u=a}^{b} s(t-u) g(u) \quad , \qquad (1)$$

where *y(t)* denotes a received signal modeled as the convolution between a source signal *s(t)* and a travel path function *g(t)*. We can write out (1) in terms of a convolution matrix and two vectors. A convolution matrix has a Toeplitz structure, which asymptotically, has the structure of a circulant matrix. We write out a trivial example where it is implicitly assumed that the length of each function is known:

$$\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} s(1) & 0 \\ s(2) & s(1) \\ s(3) & s(2) \\ 0 & s(3) \end{bmatrix} \cdot \begin{bmatrix} g(1) \\ g(2) \end{bmatrix} \qquad (2)$$

If we know two out of three functions, then we can solve for the third. However, if we only have the received signal y, no unique solution on either *s* or *g* can be made. However, given two snapshots of a source bearing signal, either from two receivers or from fractionally spaced samples of a single receiver [2], a unique solution can be made (under certain assumptions). We will suppose that there are two snapshots taken at the same time from two receivers. Both snapshots are modeled as being composed of the same signal but corrupted with different (do not share common zeros in the z-domain) multipath functions. This is written as:

$$y_1(t) = s(t) * g_1(t) \; and \; y_2(t) = s(t) * g_2(t). \qquad (3)$$

Now by convolving the first receiver with multipath function two and the second receiver with multipath function one the following equations are formed:

$$y_1(t) * g_2(t) = s(t) * g_1(t) * g_2(t) \text{ and } \qquad (4)$$
$$y_2(t) * g_1(t) = s(t) * g_2(t) * g_1(t)$$

that can be summarized as

$$y_2(t) * g_1(t) - y_1(t) * g_2(t) = 0 \quad . \qquad (5)$$

Now replace the convolution by matrix multiplication using the convolution matrices of the data and write the g_i's as vectors. Then (5) becomes

$$\begin{bmatrix} Y_2 & -Y_1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = [0] \quad , \qquad (6)$$

where Y_i is the convolution matrix of y_i and the matrix $M = [\, Y_2 \quad - Y_1 \,]$ is a (1x2) block matrix. Again, we will write this equation out in terms of vectors and matrices but here we will pad each g_i with one zero and make some observations. As an example we write out the case for a two dimensional nullspace where the subscript (i,j) refers to the i^{th} receiver and the j[th] sample, the true length of each multipath function, g_i, is two samples long, and each received signal, y_i, is four samples long.

$$\begin{bmatrix} y_{21} & 0 & 0 & -y_{11} & 0 & 0 \\ y_{22} & y_{21} & 0 & -y_{12} & -y_{11} & 0 \\ y_{23} & y_{22} & y_{21} & -y_{13} & -y_{12} & -y_{11} \\ y_{24} & y_{23} & y_{22} & -y_{14} & -y_{13} & -y_{12} \\ 0 & y_{24} & y_{23} & 0 & -y_{14} & -y_{13} \\ 0 & 0 & y_{24} & 0 & 0 & -y_{14} \end{bmatrix} \cdot \begin{bmatrix} g_{11} \\ g_{12} \\ 0 \\ g_{21} \\ g_{22} \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} y_{21} & 0 & 0 & -y_{11} & 0 & 0 \\ y_{22} & y_{21} & 0 & -y_{12} & -y_{11} & 0 \\ y_{23} & y_{22} & y_{21} & -y_{13} & -y_{12} & -y_{11} \\ y_{24} & y_{23} & y_{22} & -y_{14} & -y_{13} & -y_{12} \\ 0 & y_{24} & y_{23} & 0 & -y_{14} & -y_{13} \\ 0 & 0 & y_{24} & 0 & 0 & -y_{14} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ g_{11} \\ g_{12} \\ 0 \\ g_{21} \\ g_{22} \end{bmatrix} = [0] \qquad (7)$$

Notice that under the shifting property the two different ***g*** vectors give the same result after matrix multiplication. Clearly, each ***g*** is linearly independent of the other. This says that instead of a one dimensional nullspace which corresponds to the multipath function (up a phase shift) we have a two dimensional nullspace. Thus we observe that each time we add a column to each of the convolution matrices (and increase the nullspace dimension) we can add a zero to either end of the g_i's. Clearly, we can tie this constructed basis to the basis generated by the eigenspace decomposition via a transformation.

Noting that (6) is an equation set to zero, simply multiplying each side of the equation by the transpose of the data matrix will provide a square symmetric matrix which is block Toeplitz (asymptotically Circulant.) Now consider $Z = M^T M$. This 2x2 matrix has the following form and is explored in section 4:

$$Z = \begin{bmatrix} Y_2^T Y_2 & -Y_2^T Y_1 \\ -Y_1^T Y_2 & Y_1^T Y_1 \end{bmatrix}. \qquad (8)$$

4. SIMULATION WITH THEORTICAL IMPLICATIONS

Two snapshots are shown in Figures 1a and 1b which form the M matrix in Figure 1c. The length of each snapshot is 108, resulting from the linear convolution of a 100 sample thirty hertz source signal (dt=0.01 seconds between samples) with two uncorrelated and sparse multipath functions of length 9 (e.g. [1 0 1 0 -1 1 0 0 1] and [1 1 0 0 1 0 -1 0 1]). The number of columns for each Y_i is set at 50, which is equivalent to padding each g_i with 41 zeros. Figure 1d represents the Z matrix where the four quadrants are easily seen and have dimension 50 x 50.

Figure 2 shows the eigendecomposition of Z. In Figure 2a the 100 eigenvalues are plotted. The eigenvalues are normalized with a log scale. Slight numerical errors are show up on the eigenvalues which are slightly negative (which in theory, for symmetric matrices, should not be so). The first 42 eigenvalues represent the set of (identically equal to) zero eigenvalues indicative of the nullspace associated with the padded **g** functions which make up the multipath space. It is at this point many research papers point out that reducing the number of columns of each Y_i by 41 would leave us with a one dimensional nullspace (42-41=1) and hence two nine dimensional multipath functions (50-41 =9 and which is a stacked 18 dimensional vector in the solution space). This is fine in theory. However, by looking at the eigenvectors, is there something more to exploit in case we have something different than the ideal case? We think the answer is yes. In Figure 2b we show as an image (Matlab) the 100 columns of eigenvectors. Four quadrants are still visible and there is a division that can be seen about the 42[nd] column. It appears that the EVD shifted the four quadrant division to the left. Using the ideas of the circulant matrix, a column by column DFT of this matrix should indicate the structure of the frequency basis functions scaled by the corresponding eigenvalues. Figure 2c is the DFT (technically the spectrum) of Figure 2b. The x axis represents the eigenvector number and the y axis represents the frequency in Hz. Recall that df =1 Hz by design and 50 Hz represents Nyquist. For example, looking at column number 100 we see two points of two vertexes occurring along the y axis at 30 Hz and 70 Hz, the (positive and negative) frequencies associated with the (principal component) source signal. There is a distinct partition at eigenvector 42. The vectors to the left seem to exhibit the same DFT for each vector in the multipath space. All frequencies show up like the DFT in a noise signal. On the other hand, to the right of eigenvector 42 there is a (transposed) 2-D image of what appears to be a 1-D DFT of the concatenation of the received signals, y_1 and y_2. The vectors on the right side give what we expect from a

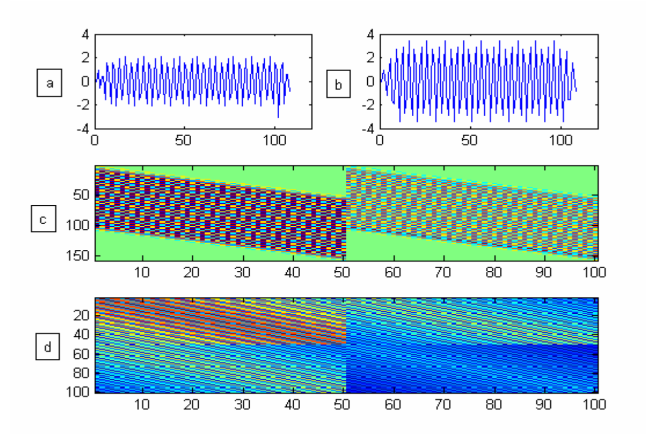

Fig. 1. (a-b) The two snap shots. (c) The M matrix of equation 6. (d) The Z matrix of equation 8.

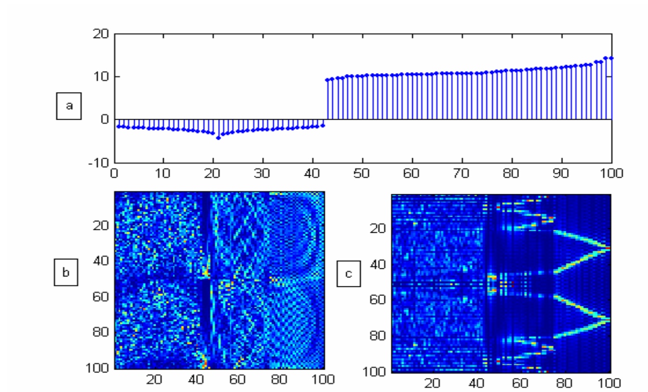

Fig. 2. (a) The set of eigenvalues normalized and ordered smallest to greatest. (b) Imagesc (Matlab) view of 100 columns of ordered eigenvectors corresponding to the eigenvalues. (c) Imagesc of the DFT (spectrum) of each column vector.

circulant matrix formed from one of the received signals. A reason the left side appears as it does can come from group theory and the idea of closure; elements from the same subgroup share a common characteristic.

Figure 3 is produced by adding a small DC offset to both y_1 and y_2 before producing the M matrix. The off diagonal zeros remain zero while each y_{ij} is increased by this offset. This has the effect of making sure there are not any zero eigenvalues. This can be seen by the smoothing out of the eigenvalues in Figure 3a. (Adding noise to each time series has a similar effect on the multipath space). There is no longer a set of (equal) zero eigenvalues. Notice that there is no longer an indication of the break at the 42nd eigenvalue. However, the vectors still show a partition in both the time frequency domains (Figures 3a and 3b). The source signal subspaces (right sides) from both figures (2b and 3b) are the same. However, with respect to the DFT of the multipath spaces from Figures 2b and 3b, there is a dramatic difference. The multipath portion of the image in Figure 3b exhibits a 'nice' frequency response. It seems that with unequal eigenvalues the DFT is allowed to show some underlying frequency response.

Furthermore, realizing the significance on how the concatenated **g** basis vectors are formed it is further reasonable to study the four quadrants separately. Recall that each column in the nullspace is related to a concatenation of a basis function from multipath function one and multipath function two. Apart from some orthogonal transformation (we conjecture) the top left quadrant represents the multipath associated with channel one and likewise the bottom left quadrant is associated with the multipath of channel 2. Like before we plot the column wise DFT of these quadrants in image form. These are seen in Figures 4a and 4b.Note that we had to pad each column with 50 zeros before performing the DFT in order to maintain a consistent df = 1 Hz and Nyquist at 50 Hz. The images in 4c and 4d represent the filtered source signal subspaces, for which channels we are not sure at this time.

5. CONCLUSIONS

In this paper we have shown that the eigenvectors hold information in the frequency domain either with or without noise. The four quadrants seem to be an artifact from the product of the matrix and its transpose. Interestingly the dimensions of the quadrants within the Z matrix change in the eigenvector representation reflecting the calculation of a multipath space. We have conjectured that since we are using circulant type matrices there should be some relation between eigenvectors and the DFT and that it may be useful to apply this idea to the so called nullspace especially in the presence of a DC offset or noise. A detailed look at each quadrant may shed some light on the characteristics of the channel response. In any event, using the information in the eigenvectors may help to determine channel order when subspace algorithms using eigenvalues can no longer show the break between spaces.

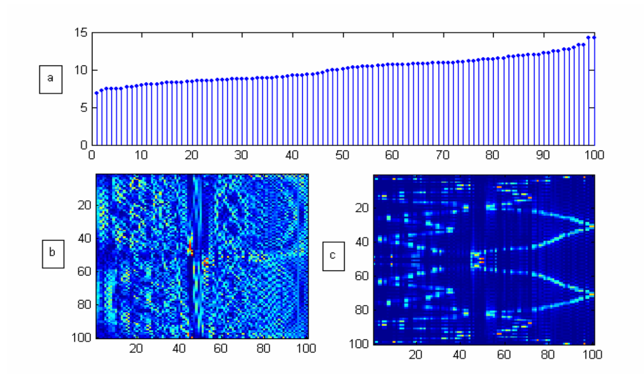

Fig. 3. The EVD after adding a DC offset. (a) The set of eigenvalues normalized and ordered smallest to greatest after adding dc offset. (b) Imagesc (Matlab) view of 100 columns of ordered eigenvectors corresponding to the eigenvalues. (c) Imagesc of the DFT (spectrum) of each column vector.

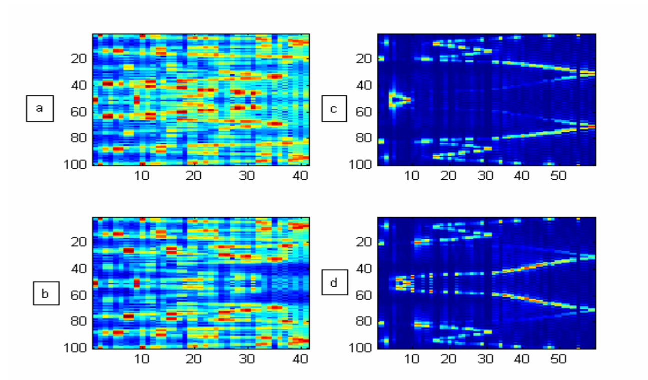

Fig. 4. DFT Images of individual quadrants. (a) Channel one multipath. (b) Channel two multipath. (c) Source Signal subspace. (d) Source signal subspace.

5. REFERENCES

[1] R. Weber, A. Waldhorst, F. Schulz, and J.F. Bohme, "Channel Order Estimation: A Comparative Study", 10th Aachen Symposium on Signal Theory, Aachen, Germany, Sept. 2001.
[2] D.T.M. Slock and C.B. Papadias, "Further results on blind identification and equalization of multiple FIR channels," in IEEE Proc. ICASSP, Detroit, MI, p. 1964-1967, Apr. 1995.
[3] Eike Rietsch, "Euclid and the art of wavelet estimation, Part I", Geophysics, vol. 62, p. 1931-1938, Nov. /Dec. 1997.
[4] Todd K. Moon and Wynn C. Stirling, *Mathematical Methods and Algorithms for Signal Processing,* Upper Saddle River, NJ, Prentice Hall, 2000.
[5] James P. LaRue, "Channel order selection in blind deconvolution", Tri-Service Waveform Diversity Workshop, February, 2004, Turning Stone Casinao, Vernon, NY.
[6] Ruifeng Zhang, "Blind Channel Estimation for Prerecorded OFDM System", in IEEE Proc. ICASSP, Philadelphia, PA, p. 469-472, March, 2005.