Singular Value Decomposition Analysis for Two Channel Signal Separation

Shortened Title

SVD Analysis for Channel Separation

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Abstract

This paper addresses the problem of separating a source signal from its multipath when the signal is unknown.

Many signal processing methods, including Blind Deconvolution and Blind Signal Separation, designed to help the separation process require knowledge of the filter length. In these methods the filter length is usually estimated from a function of eigenvalues of the multichannel matrix associated with the underlying process. While these methods work well in theory, they often break down in the presence of noise.

Instead of looking at the eigenvalues, we propose to estimate the filter length from the null space, also referred to as the multipath subspace, of the multichannel matrix. The multipath subspace is spanned by vectors which exhibit circular structure. This structure aids in determination of the filter length.

This paper focuses on the two channel separation problem. A computational algorithm is proposed for joint order detection and channel estimation from the structured basis of the multipath subspace. In low signal-to-noise environments its performance compares favorably with the performance of algorithms which are based on the distribution of eigenvalues only.

1 Introduction

Multipath is a term that applies to a signal that has been corrupted by either advanced or delayed copies of itself. Signals contaminated by multipath are modeled as a convolution process which in the discrete time domain can be expressed as

$$y(n) = \sum_{i=0}^{L-1} s(n-i)g(i) + v(n)$$

where $y(k)$ denotes the kth sample of the received signal, $s(k)$ is the the kth sample of the source signal, $v(n)$ represents additive noise, and $g(k)$'s describes the channel of length L.

In many applications neither s nor g are known and hence signal processing techniques for estimating s and g require either multiple receivers or fractionally spaced samples in the case of a single receiver. In any case, the input data to a signal processor consists of $M > 1$ sequences of N samples each. This is illustrated in Figure 1.

It is known that in the noiseless case, $v_i(n) = 0$, under some mild assumptions, channels $\{g_m(l)\}_{m=1,l=0}^{M,L_m-1}$ can be uniquely identified from $\{y_m(n)\}_{m=1,n=1}^{M,N}$, [6], [8].

In the past the first step in the blind channel estimation problem was to determine the size of the multipath subspace associated with channel equalization. In the event that the dimension of the multipath subspace was larger than one the dimension of the multichannel matrix had to be resized in order to produce the one-dimensional multipath subspace. Research suggested that only then could the multipath be identified.

However in this paper we show that in the case of the multidimensional multipath subspace it is not necessary to first reduce its dimension to find the multipath solution. Rather, the desired multipath solution can be identified directly from the basis of the subspace spanned by a set of circularly shifted vectors.

In this paper we show how this circularly shifted basis vectors can be achieved from multidimensional multipath subspace by a sequence of elementary linear transformations. In the process of doing that the filter length and filter coefficients can be easily identified. This method works well in the presence of AWGN noise and performs favorably against eigenvalue based methods and is an improvement over the method that contrasted characteristics of singular vectors spanning the signal and null spaces which was proposed in [4].

We will concentrate on the case of two receivers although our derivations can be directly extended to the case when there are more receivers.

2 Joint order and channel estimation from singular values

Suppose that there are two snapshots $y_1(n)$ and $y_2(n)$ taken at the same time from two receivers. Both snapshots are modeled as being composed of the same signal but corrupted with different multipath functions. The two snapshots can be expressed as convolution of signal s with the multipath channels g_1 and g_2,

$$y_1(n) = s(n) * g_1(n) \quad \text{and} \quad y_2(n) = s(n) * g_2(n)$$

where '*' denotes the convolution operation.

By convolving each of the received signals with the channel function of the other received signal we obtain

$$y_1(n) * g_2(n) = y_2(n) * g_1(n) \tag{1}$$

Assume that N samples of the two discrete-time signals were recorded. Let y_1 and y_2 be vectors composed from samples, $y_1 = \{y_{j,1}\}_{j=1}^{N}$ and $y_2 = \{y_{j,2}\}_{j=1}^{N}$ where $y_{j,i} = y_i(j)$. Then (1) can be expressed in the matrix form (2)

$$Y \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = 0 \tag{2}$$

where $Y = (Y_2 \quad -Y_1)$, $g_i = \{g_{j,1}\}_{j=1}^{L_i}$ is a vector of channel coefficients, and Y_i is the convolution matrix of y_i,

$$Y_i = \begin{pmatrix} y_{1i} & & & \\ y_{2i} & y_{1i} & & \\ \vdots & & \ddots & \\ y_{ni} & & & y_{1i} \\ \vdots & & & \vdots \\ y_{Ni} & & & y_{N-L_i+1,i} \\ & y_{N1} & & y_{N-L_i+2,i} \\ & & \ddots & \vdots \\ & & & y_{Ni} \end{pmatrix}, \quad g_i = \begin{pmatrix} g_{1i} \\ g_{2i} \\ \vdots \\ g_{L_i i} \end{pmatrix} \tag{3}$$

Here, the length of channel g_i is L_i and hence the matrices Y_i have dimensions $(N + L_i - 1) \times L_i$, $i = 1, 2$. We will assume that $L_1 = L_2 = L$ (this can be assured by padding the shorter channel with zeros).

It can be shown [8] that in the noiseless case under some mild assumptions the matrix Y has one dimensional nullspace. In (2) the nullspace is determined by a vector which is formed by concatenating the two multipath channel vectors g_1 and g_2. Equation (2) allows for the calculation of (a scaled version of) g by finding the one dimensional nullspace of Y. For that reason we will refer to this nullspace as a multipath subspace.

2.1 Noisless case

In practice the length of the multipath is not know. A general way to recover g from Y is to overestimate the length of g_1 and g_2, collect delayed copies of the signals y_1 and y_2 into a block Toeplitz matrix $Y^{(ext)}$ (which is defined in the next Section), calculate the dimension of its nullspace, and next reduce the number of columns in $Y^{(ext)}$ so the resulting matrix is the desired matrix Y in (2), [6], [8]. This one dimensional nullspace is spanned by the right singular vector of Y corresponding to its zero singular value. More precisely, the process proceeds as follows.

Say that the extended matrix $Y^{(q)} = \left(Y_2^{(q)} \; -Y_1^{(q)} \right)$ has $p = 2(L+q-1)$ columns, $q \geq 1$. One can determine the dimension of the nullspace of $Y^{(q)}$ from its singular value decomposition (SVD),

$$U \Sigma V^H = Y^{(q)}.$$

If there are q zero singular values then removing the last $q-1$ columns from each of the matrices $Y_1^{(q)}$ and $Y_2^{(q)}$ results in the desired matrix Y.

2.2 Noisy case

In the presence of noise the received data vectors $\hat{y}_1$ and $\hat{y}_2$ are perturbed versions of the true noiseless vectors y_1 and y_2. Thus we have

$$\hat{y}_1 = y_1 + e_1 \ , \ \hat{y}_2 = y_2 + e_2$$

where e_i represent uncorrelated noise evectors. Thus the data matrix $\hat{Y}^{(q)}$ satisfies $\hat{Y}^{(q)} = Y^{(q)} + E$ where E is a block Toeplitz matrix constructed from the noise vectors e_1 and e_2. It can be shown that $\hat{Y}^{(q)}$ is almost always full rank. Thus one cannot speak about its nullspace but rather one wants to identify a perturbed multipath subspace, which we will still refer to as the multipath subspace. This subspace can be determined by identifying a gap among singular values. Let $\hat{\sigma}_i$ denote the singular values of $\hat{Y}^{(q)}$. If singular values satisfy the inequality

$$\hat{\sigma}_1 \geq \cdots \geq \hat{\sigma}_{p-q} >> \hat{\sigma}_{p-q+1} \geq \cdots \geq \hat{\sigma}_p.$$

we say that there is a gap (or break) in the spectrum between $\hat{\sigma}_{p-q}$ and $\hat{\sigma}_{p-q+1}$. Now the right singular vectors of $\hat{Y}^{(q)}$ corresponding to singular values to the right of the gap span the multipath subspace of $\hat{Y}^{(q)}$. The gap allows for the estimation of the dimension of the nullspace of noise free $Y^{(q)}$. Once the dimension is established, redundant columns of $\hat{Y}^{(q)}$ are removed resulting in $\hat{Y}^{(1)}$. The approximate channel $\hat{g}$ is obtained as the right singular vector v_{2L} of $\hat{Y}^{(1)}$ associated with its smallest singular value $\hat{\sigma}_{2L}$.

There is a number of techniques to find the gap in the spectrum. Among others the most often mantioned are

(AIC) Akaike Information Criterion [1], [10], selects k which minimizes

$$AIC(k) = -2(p-k)N \log \frac{\mathcal{G}(\hat{\sigma}_{k+1}, ..., \hat{\sigma}_p)}{\mathcal{A}(\hat{\sigma}_{k+1}, ..., \hat{\sigma}_p)} + 2k(2p-k)$$

where $\mathcal{G}$ and $\mathcal{A}$ are the geometric and arithmetic means, respectively.

(MDL) Minimum Description Length [7], [10], select k which minimizes

$$MDL(k) = -(p-k)N \log \frac{\mathcal{G}(\hat{\sigma}_{k+1}, ..., \hat{\sigma}_p)}{\mathcal{A}(\hat{\sigma}_{k+1}, ..., \hat{\sigma}_p)} + \frac{1}{2}k(2p-k)\log N$$

where $\mathcal{G}$ and $\mathcal{A}$ are the geometric and arithmetic means, respectively.

(CA) Canonical angles [5] criterion select k which minimizes

$$ca(k) = \begin{cases} \frac{\hat{\sigma}_{k+1}^2}{\hat{\sigma}_k^2 - 2\hat{\sigma}_{k+1}^2} & \text{if } \sqrt{3}\hat{\sigma}_{k+1} \leq \hat{\sigma}_k \\ 1 & \text{otherwise} \end{cases}$$

(GAP) Singular values ratio test selects k which maximizes $sr(k) = \frac{\hat{\sigma}_k}{\hat{\sigma}_{k+1}}$

(DR) Difference of ratios test [9] selects k which maximizes $dr(k)$,

$$dr(k) = \frac{\hat{\sigma}_{k+2}}{\hat{\sigma}_{k+1}} - \frac{\hat{\sigma}_{k+1}}{\hat{\sigma}_k}$$

Some of these techniques were evaluated in [5]. It was shown in [5] that estimates of the dimension of the nullspace obtained from the distribution of eigenvalues (or singular values) are often inaccurate. This is caused by the presence of noise and small number of available samples. For example, a typical performance of the GAP test is illustrated in Figure 2.

When the signal-to-noise ratio SNR is high, Figure 2(a), the GAP test correctly estimates the dimension of the nullspace. For low SNR, Figure 2(b), the test fails completely. This happens because singular values of the noisy data matrix can move as much as the norm $||E||$ of the noise matrix E. Thus if the largest ratio of consecutive singular values for noise free data is found among small singular values then addition of noise can significantly decrease this ratio.

3 Joint order and channel estimation from singular vectors

Instead of, or in addition to, trying to identify a gap between singular values one can explore the information embedded in the right singular vectors. The reason for this is that the matrix Y is block Toeplitz.

3.1 Noisless case

Consider first the noiseless case. Suppose that the system matrix $Y^{(1)} = Y$ has one dimensional null space and that $g^{(1)} = [g_1^T, g_2^T]^T$ is its basis vector, as illustrated in (4),

$$Y^{(1)}g^{(1)} = \left(\begin{array}{ccc|ccc} y_{21} & 0 & 0 & -y_{11} & 0 & 0 \\ y_{22} & y_{21} & 0 & -y_{12} & -y_{11} & 0 \\ y_{23} & y_{22} & y_{21} & -y_{13} & -y_{12} & -y_{11} \\ 0 & y_{23} & y_{22} & 0 & -y_{13} & -y_{12} \\ 0 & 0 & y_{23} & 0 & 0 & -y_{13} \end{array}\right) \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = 0 \qquad (4)$$

Suppose a zero is appended to each g_i and define $g^{(2)}$ as the concatenation of such extended vectors, $g^{(2)} = [g_1^T\ , 0\ , g_2^T\ , 0]^T$. Suppose further that the matrix $Y^{(1)}$ is extended to $Y^{(2)}$ by adding to it a zero row and next adding circularly shifted copies of extended by zeros vectors y_1 and of y_2, see (5),

$$Y^{(2)} = \left(\begin{array}{cccc|cccc} y_{21} & 0 & 0 & 0 & -y_{11} & 0 & 0 & 0 \\ y_{22} & y_{21} & 0 & 0 & -y_{12} & -y_{11} & 0 & 0 \\ y_{23} & y_{22} & y_{21} & 0 & -y_{13} & -y_{12} & -y_{11} & 0 \\ 0 & y_{23} & y_{22} & y_{21} & 0 & -y_{13} & -y_{12} & -y_{11} \\ 0 & 0 & y_{23} & y_{22} & 0 & 0 & -y_{13} & -y_{12} \\ 0 & 0 & 0 & y_{23} & 0 & 0 & 0 & -y_{13} \end{array}\right). \tag{5}$$

It has been shown in [6] that the nullspace of $Y^{(2)}$ is spanned by two vector, $g^{(2)}$ and and its circularly shifted copy $[0\ , g_1^T\ , 0\ , g_2^T]^T$,

$$Y^{(2)} \begin{pmatrix} g_1 & 0 \\ 0 & g_1 \\ g_2 & 0 \\ 0 & g_2 \end{pmatrix} = Y_2^{(2)} G_2 = 0.$$

In analogous way we can create matrix $Y^{(q)}$ by adding $(q-1)$ zero rows to $Y^{(1)}$ and next by appending such obtained matrix with $q-1$ circularly shifted copies of extended with zeros vectors y_i. The nullspace of $Y^{(q)}$ will be spanned by the vector $g^{(q)} = [g_1^T\ , 0, \ldots, g_2^T\ , 0, \ldots, 0]^T$ and $q-1$ of its circularly shifted copies. It is convenient to collect all these shifted vectors into the matrix G_q. The space spanned by columns of G_q the multipath subspace. We will say that columns of G_q form the *fundamental* basis for the multipath subspace of $Y^{(q)}$.

Recall that this multipath subspace is also spanned by the q right singular vectors $V_q = [v_{p-q+1}, ..., v_p]$ of $Y^{(q)}$ corresponding to the q zero singular values of $Y^{(q)}$.

Figure 3 illustrates these unstructured and structure basis for the nullspace of $Y^{(q)}$.

As columns of V_q, Figure 3(a), and columns of G_q, Figure 3(b), span the same subspace, there exists a nonsigular matrix L such that, [3]

$$V_q L = G_q.$$

That matrix L can be found by first postmultiplying V_q by an orthogonal transformation Q_q which transforms V_q into an upper trapezoidal matrix $W_q = V_q Q_q$. The next step is to apply Gaussian elimination to columns of W_q to zero all elements above the main diagonal. This process is illustrated in Figure 4.

Let the Gaussian elimination matrix be R_q. Then the elimination process results in a matrix F_q which has the structure shown in (6)

$$F_q = V_q Q_q R_q = \left(\begin{array}{cccc} f_1 & 0 & \cdots & 0 \\ \times & f_2 & & \vdots \\ \vdots & & \ddots & \\ \times & & \times & f_q \\ \hline h_1 & \times & & \times \\ \vdots & h_2 & & \vdots \\ & \vdots & \ddots & \times \\ 0 & 0 & 0 & h_q \end{array} \right) \tag{6}$$

where f_i and h_i are vectors of the same dimension and $\times$'s are some other entries.

Under mild conditions [6], [8] one can show that the fundamental basis G_q is unique and hence we must have that $F_q = G_q$. Thus $f_1 = \ldots = f_q = g_1$, $h_1 = \ldots = h_q = g_2$, and $\times$'s are all zeros. This observation allows us to find the channel length and channel coefficients when the dimension q of the nullspace is not know. A possible strategy is described in next.

Remark. The fact that the multipath subspace has a structured basis was also exploited in [2] in the context of MIMO communication channels. However there the SVD is not computed but rather a Levinson-type algorithm is used in conjunction with the finite alphabet property of the signals.

3.2 Noisy case

We now consider the case when the received signal is contaminated by noise. Suppose we collected N samples of the received signals $\hat{y}_1$ and $\hat{y}_2$ and constructed a $(N+k-1)\times 2k$ data matrix $\hat{Y}$ as in (5). Suppose that $\hat{Y} = \hat{Y}^{(q)}$ for some $q \in [D_{min}\,,\ D_{max}]$ with D_{min} and D_{max} known. Thus $\hat{Y}$ has a multipath subspace of dimension q. We would like to find that dimension q.

The first step is to compute the SVD decomposition of $\hat{Y}$, $\hat{Y} = U\Sigma V^H$ where $V = [v_1\,,\ v_2\,,\ \ldots\,,\ v_{2k}]$. Now we consider subspaces $\mathcal{V}_i$ spanned by the columns of $V_i = [v_{2k-i+1}, \ldots, v_{2k}]$. Clearly the noise subspace is spanned by the columns of one of these V_i's.

Say that V_i for $i = q$ spans the noise subspace. Then there should exist a basis of this subspace which when slightly perturbed will exhibit the structure of fundamental basis. Thus we want to check whether there exists a linear transformation $\hat{L}$ of V_i such that

$$V_i L = \hat{G}^{(i)} = \left(\begin{array}{cccc} \hat{f}_1 & 0 & \cdots & 0 \\ \times & \hat{f}_2 & & \vdots \\ \vdots & & \ddots & \\ \times & \cdots & \times & \hat{f}_i \\ \hline \hat{h}_1 & \times & & \times \\ \vdots & \hat{h}_2 & & \vdots \\ & \vdots & \ddots & \times \\ 0 & 0 & 0 & \hat{h}_i \end{array}\right) \tag{7}$$

where all $\hat{f}_j$ are approximately the same, all $\hat{h}_j$ are approximately the same, and elements denoted by $\times$ are small relatively to $\hat{f}_j$'s and $\hat{h}_j$'s.

As in the case of known multipath dimension, we first postmultiply V_i by an orthogonal transformation Q_i which transforms V_i into the upper trapezoidal form W_i. Next we apply Gaussian elimination to columns of W_i to zero all elements above the main diagonal. Then we check whether the elimination results in a matrix F_i which has the structure similar to the one shown in (7).

In the noisy case neither $\hat{h}_j$'s nor $\hat{f}_j$'s will be equal. Similarly $\times$'s will not be zeros. Thus we need a criterion for selecting the most likely, in some sense, dimension i for which $\hat{G}^{(i)}$ is an approximate fundamental basis.

We propose the following two measures. In theory, all h_j' and f_j' should be equal. With the presence of noise we can check the degree of correlation among these vectors. This can be done, for example by, computing the SVD of the matrices $H^{(i)} = [\hat{h}_1, \ldots, \hat{h}_i]$ and $F^{(i)} = [\hat{f}_1, \ldots, \hat{f}_i]$. The index i is then selected as the one for each the ratio $sd_H(i)$ and $sd_F(i)$ of the largest to the second largest singular value of $H^{(i)}$ and $F^{(i)}$, respectively, is maximal.

Similarly, in theory the residual matrix $E^{(i)} = Y[(G^{(i)})^T \ \ (H^{(i)})^T]^T$ should be a zero matrix. In the presence of noise we select the index i so the residual error $e_i = ||E^{(i)}||$ is the smallest possible.

Finally, we we can select i as a function of e_i, $sd_H(i)$ and $sd_F(i)$. In numerous simulations we found out that the following heuristic criterion predicts well the dimension of the multipath subspace. Namely, we choose i which minimizes the function $\epsilon(i)$,

$$\epsilon(i) = ||E^{(i)}|| + \left(\frac{1}{sd_H(i)}\right)^{\frac{1}{2}} + \left(\frac{1}{sd_F(i)}\right)^{\frac{1}{2}},$$

over all $i \in [D_{min}\ ,\ D_{max}]$.

A typical performance of this criterion is illustrated in Figure 5 where $D_{min} = 4$, $D_{max} = 12$ and $q = 6$. There, our criteria selected the correct dimension of the multipath subspace which was $i = 6$. The basis of this multipath subspace is the one shown in Figure 5(c). Note however that the basis for $i = 9$ shown in Figure 5(f) can also be consider as a potential candidate.

Note that once q is selected, one not only have an estimate of the filter length but also an estimate of the filter coefficients. These can be obtained from the columns of the matrix $\hat{G}_q$.

In the next section we present experimental results comparing eigenvalues based methods with the method described in this section.

4 Simulations

Performance of all six methods was tested for the following synthetic data. A pure sinusoid was generated in MATLAB by the code:

```
t = (0:.001:sdur);
y = sin(2*pi*fr*t);
```

where `sdur` determined the duration of the signal and `fr` its frequency. Two multipath vector g_1 and g_2 were chosen as

```
g1 = [1 0 0 1 0 1 0 1];
g2 = [1 0 1 0 1 0 0 1];
```

Two multipath signals were obtained as

```
z1 = conv(g1,y);
z2 = conv(g2,y);
```

Next white noise was added to the multipath signals

```
y1 = awgn(z1,snr);
y2 = awgn(z2,snr);
```

where `snr = 20,24,...,32`. The data matrix Y was built from noisy multipath signals. Its size was overestimated so the corresponding noiseless multipoath space had dimension `K` where `K = 4,6,8,...,14`.

For each SNR level `snr` and for each dimension `K` of the nullspace all five eigenvalue based algorithms from Section 2.2 and the singular vectors based algorithm from Section 3.2 were executed 100 times each. The fractions of correct estimations by the algorithms are ploted in Figure 6.

Graphs in Figure 7 show the average filter length computed by different algorithms over 100 tests for each dimension of the nullspace.

The experiments suggest that the method which takes into the account the information contained in the singular vectors often outperforms the methods that take into the account the information contained in the singular values only. This is specially true in the cases when the dimension of the multipath subspace is low and signal-to-noise ratio is low as well.

5 Conclusions

Spectral techniques like SVD may fail in presence of noise. It often helps to exploit additional information presents in the structured data. In the case of the multichannel estimation problem considered in this paper the relevant structure is that of a block circulant data matrix whose nullspace is spanned by columns of a circulant matrix. We have shown that the circulant structure of the nullspace helps in determination of its dimension.

We have also shown that in the case of the multidimensional multipath subspace it is not necessary to first reduce its dimension as is done in methods based on eigenvalue information only. Rather, the desired multipath solution can be identified directly from the basis of the subspace spanned by a set of circularly shifted null vectors.

That basis was extracted from the singular vector basis of the overexteneded multipath subspace by creating a linear transformation from the singular vector basis to the fundamental basis.

We proposed a criterion for filter length estimation and designed a linear transformation from the singular vectors basis to the fundamental basis that worked well in numerous Monte Carlo simulations. However it is not clear whether our criteria are the best possible. It is plausible that our techniques can be improved. This is a subject of current investigation.

We considered the case of two receivers. However, our technique can be easily generalized to the case of any number of receivers. Adding more receivers improves estimations at the cost of more computations. The trade-offs between the quality of estimation and the cost is another subject of ongoing research.

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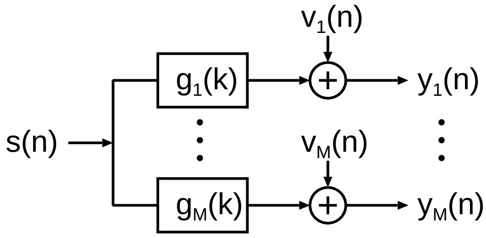
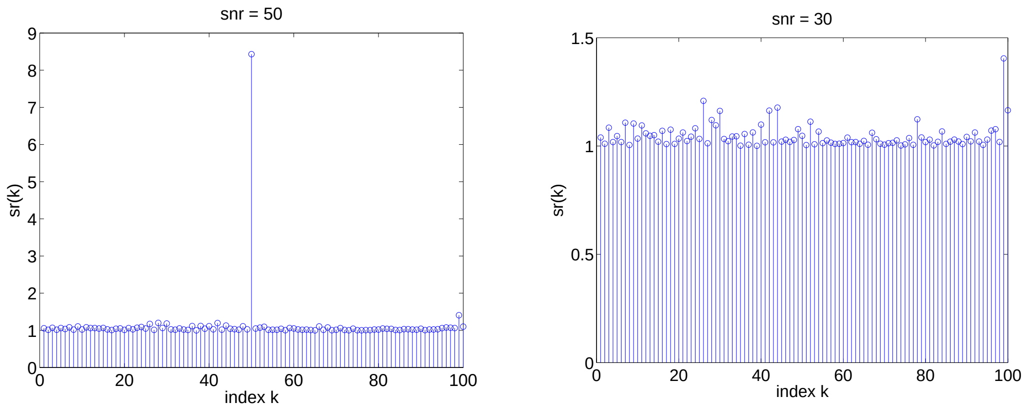
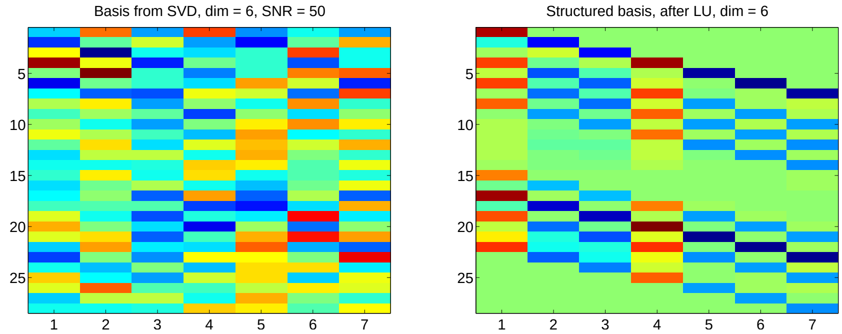


FIGURE 1. M channel receiver.



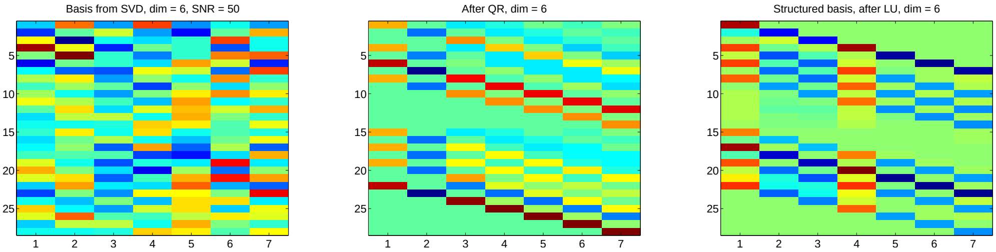
(a) SNR = 50 (b) SNR = 30

FIGURE 2. GAP test, $sr(k)$ are plotted against k.



(a) Unstructured basis (b) Structured basis

FIGURE 3. Result of MATLAB's (a) `imagesc(V)` and (b) `imagesc(G)`.



(a) Null vectors V_q (b) Null vectors W_q (c) Structured basis F_q

FIGURE 4. Basis obtained form (a) SVD, (b) after QR factorization, (c) after LU decomposition.

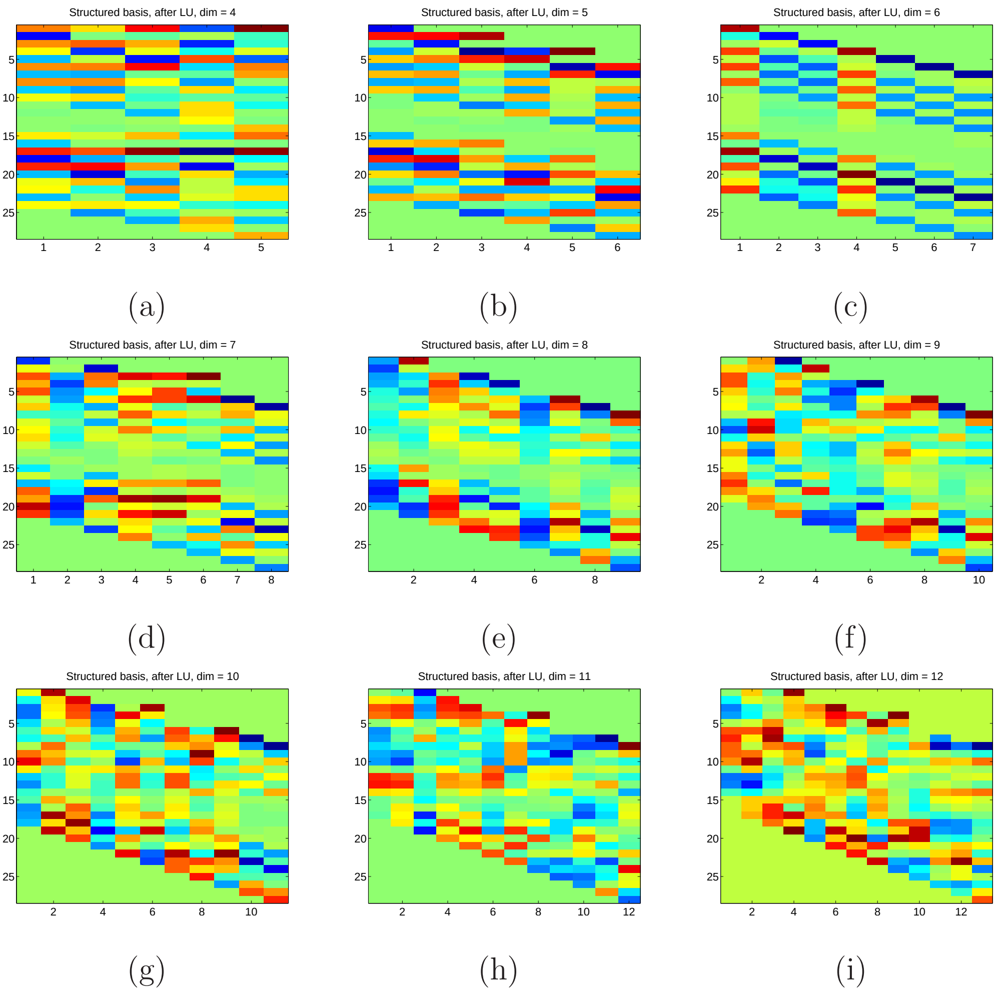


FIGURE 5. Candidate basis vectors examined by the algorithm.

$f = 330$, snr $= 20, 22, ..., 34$, duration 0.09

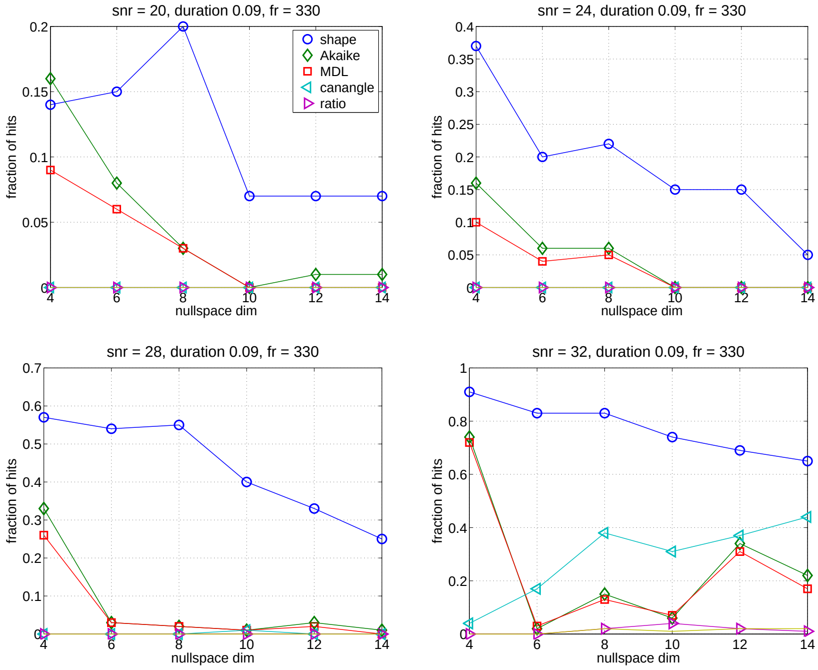


FIGURE 6. Fractions of correct filter length estimations.

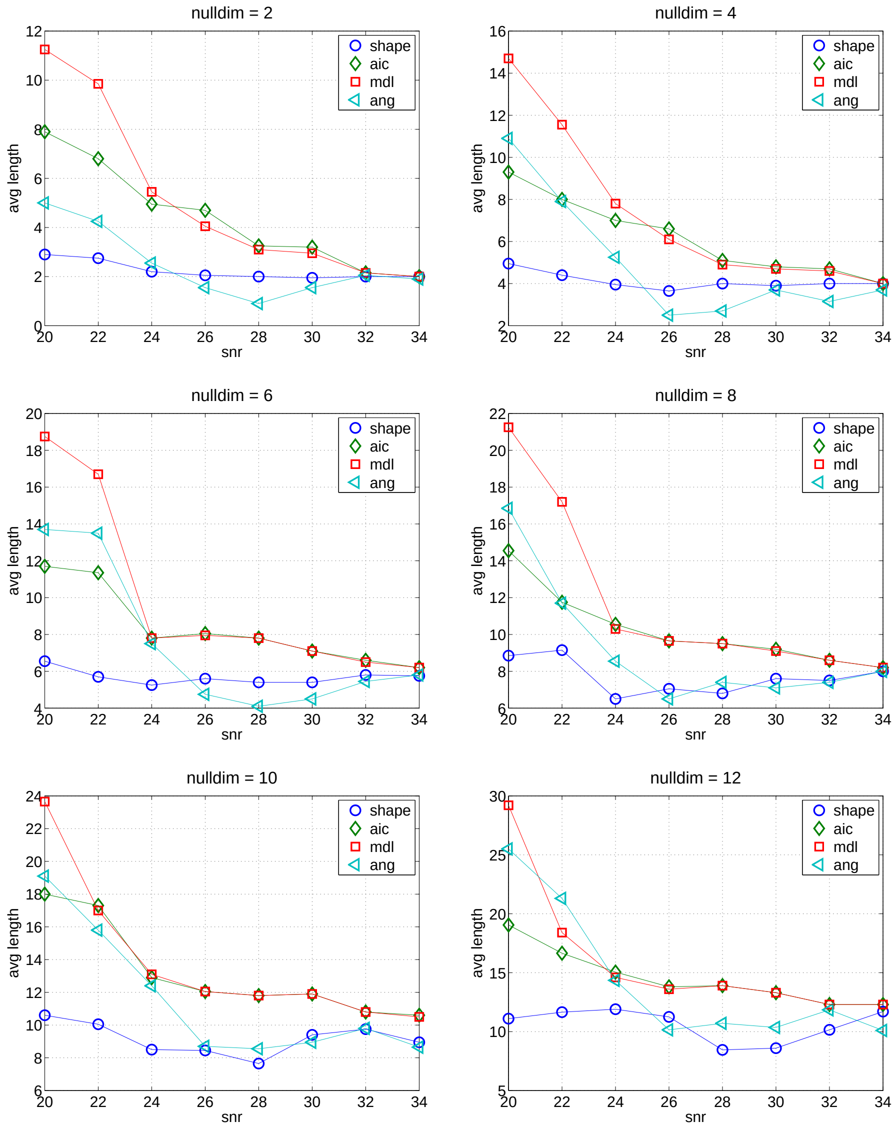


FIGURE 7. Avarage filter length.